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STATISTICAL MAGNETOHYDRODYNAMICS AND REVERSED-FIELD-PINCH QUIESCENCE*

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Abstract

A statistical model of a bounded, incompressible, cylindrical magnetofluid is presented. This model predicts the presence of magnetic fluctuations about a cylindrically-symmetric, Bessel-function-model, mean magnetic field, which satisfies $\nabla \times \langle \underset{\sim}{B} \rangle = \mu \langle \underset{\sim}{B} \rangle$. As $\Theta \to 1.56$, the model predicts that the significant region of the fluctuation spectrum narrows down to a single (coherent) $m = 1$ mode. An analogy between the Debye length of an electrostatic plasma and μ^{-1} suggests the physical validity of the model's prediction of $\langle \delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle$ when $|\underset{\sim}{r} - \underset{\sim}{r}'| > \mu^{-1}$.

1. INTRODUCTION

Certain phenomena appear to occur often in reversed-field-pinch experiments such as ZT-40M and HBTX-1A. After an initial turbulent setting-up phase, the plasma relaxes to a quieter reversed-field configuration, with a Θ value of approximately 1.5-1.6 and with low-m modal activity evident. In peripheral magnetic-field data obtained from HBTX-1A, coherent $m = 1$ activity is observed only during quiet periods.[1]

A rigorous treatment of three-dimensional turbulent dynamics of a bounded, driven, dissipative plasma is well beyond current analytical and numerical capabilities.

We shall outline in this paper a statistical model of bounded, three-dimensional, incompressible, ideal magnetohydrodynamic turbulence in an attempt to explain the salient properties of the relaxed, reversed-field-pinch configuration. Our treatment will ignore the effects of dissipation, and therefore will be unable to portray realistically the high-wavenumber spectrum or the equivalent short-range correlations. Our model[2] is rooted in Taylor's pioneering work on the reversed-field pinch[3] as well as in our earlier collaborations with Montgomery[4] and Christiansen.[5]

Although this paper confines itself to heuristic reasoning, a much lengthier, mathematically more rigorous paper is in preparation.[6]

2. THE ALGORITHM

The fields necessary to the specification of the state of a system's configuration are expanded in terms of a complete set of states, the amplitude of each state being specified by a time-dependent spectral coefficient, $c_i(t)$. When one inserts such an expansion into the governing dynamical equations (assumed to be ideal; i.e., lossless), one obtains an infinite-dimensional representation of the equations; that is, an infinite set of equations, generally nonlinear, specifying the time-rate-of-change of each coefficient as a function of the current values of all the coefficients. Instead of attempting an actual solution of these equations, we instead shall follow a path suggested originally by Lee.[7]

We first ascertain that $\sum_i \partial\dot{c}_i(t)/\partial c_i(t) = 0$, thereby demonstrating the validity of a Liouville theorem in the infinite-dimensional "phase space" spanned by the linearly independent components of the spectral coefficients. Therefore an ensemble of points (each point uniquely specifying a unique configuration of the system) flows incompressibly in this "phase space." We then identify the quadratic invariants of the ideal equations, such as the kinetic energy and the enstrophy of a two-dimensional, incompressible Euler fluid, or the total energy, the cross-helicity, and the magnetic helicity of a three-dimensional, incompressible magnetofluid. Postulating equal *a priori* probability of finding the system in any region of the "phase space" allowed by the constraints imposed by the quadratic invariants, we invoke the machinery of classical statistical mechanics to derive the canonical distribution for the absolute equilibrium ensemble. This distribution is the normalized exponentiation of the linear superposition of the quadratic invariants, each with its own Lagrange multiplier (inverse "temperature") and each expressed in terms of the spectral coefficients. The mean value of any physical quantity of the system, which is a function of these coefficients, is then assumed to be the ensemble-averaged value.

3. APPLICATION TO A TWO-DIMENSIONAL EULER FLUID

We shall demonstrate the use of our algorithm in the context of a uniform-density (taken to be unity), two-dimensional, periodic, incompressible Euler fluid. This fluid is described by:

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = -\nabla p, \qquad \nabla\cdot\underset{\sim}{u} = 0,$$

in which the fluid velocity, $\underset{\sim}{u}(\underset{\sim}{r},t)$, is in the x-y plane and both it and the pressure, $p(\underset{\sim}{r},t)$, are functions of only x and y. By virtue of the periodicity, the velocity field has the Fourier expansion:

$$\underset{\sim}{u}(\underset{\sim}{r},t) = \sum_{\underset{\sim}{k}} \tilde{\underset{\sim}{u}}(\underset{\sim}{k},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{r}).$$

One readily can demonstrate the conservation of the energy,

$$\varepsilon \equiv \int u^2 \frac{d^2r}{\Delta},$$

and of all the moments of the vorticity,

$$\Omega^{(n)} \equiv \int \omega^n \frac{d^2r}{\Delta},$$

where

$$\omega(\underset{\sim}{r},t) = \sum_{\underset{\sim}{k}} \tilde{\omega}(\underset{\sim}{k},t)\exp(ik\cdot r) \equiv \hat{z}\cdot\nabla \times \underset{\sim}{u}(\underset{\sim}{r},t). \tag{1}$$

The domain of integration is the unit periodic cell of area, Δ. One notes that

$$\tilde{\underset{\sim}{u}}(\underset{\sim}{k},t) = \frac{i\underset{\sim}{k} \times \hat{z}}{k^2}\tilde{\omega}(\underset{\sim}{k},t).$$

Reality of the vorticity requires that

$$\tilde{\omega}(\underset{\sim}{k},t) = \tilde{\omega}^*(-\underset{\sim}{k},t). \tag{2}$$

Conservation of $\Omega^{(1)}$ allows us to choose $\tilde{\omega}(0,t) = 0$. One easily can confirm the validity of the Liouville theorem in the infinite-dimensional "phase space" spanned by the linearly independent, real and imaginary parts of the spectral (Fourier) coefficients, $\tilde{\omega}(\underset{\sim}{k},t)$. Retaining the constraints of only the quadratic invariants, energy and enstrophy, in the prescription of the absolute equilibrium ensemble distribution, D_{eq}, we find:

$$D_{eq} \propto \exp[-\beta(\Omega^{(2)} + \kappa^2\varepsilon)] = \exp[-\beta\sum_{\underset{\sim}{k}}(\frac{\kappa^2 + k^2}{k^2})|\tilde{\omega}(\underset{\sim}{k})|^2],$$

where the inverse "temperatures" associated with the enstrophy and energy are β and $\kappa^2\beta$, respectively, and where the enstrophy and energy have the respective Fourier expansions:

$$\Omega^{(2)} = \sum_{\underset{\sim}{k}} |\tilde{\omega}(\underset{\sim}{k})|^2,$$

$$\varepsilon = \sum_{\underset{\sim}{k}} \frac{|\tilde{\omega}(\underset{\sim}{k})|^2}{k^2}.$$

Using the reality condition for the vorticity, eq. (2), one can verify that the mean value of $\tilde{\omega}(\underset{\sim}{k})\tilde{\omega}(\underset{\sim}{k}')$ for this ensemble is given by:

$$\langle\tilde{\omega}(\underset{\sim}{k})\tilde{\omega}(\underset{\sim}{k}')\rangle = \delta_{\underset{\sim}{k},-\underset{\sim}{k}'} \left[\frac{k^2}{2\beta(k^2 + \kappa^2)}\right], \tag{3}$$

where the Kronecker delta equals one if $\underset{\sim}{k} + \underset{\sim}{k}' = 0$, and equals zero otherwise. A direct consequence of eqs. (1) and (3) is the autocorrelation function of the vorticity:

$$\langle\omega(\underset{\sim}{r})\omega(\underset{\sim}{r}')\rangle = \frac{1}{2\beta} \sum_{\underset{\sim}{k}} \frac{k^2}{k^2 + \kappa^2} \exp[i\underset{\sim}{k}\cdot(\underset{\sim}{r} - \underset{\sim}{r}')].$$

If one lets the periodic cell become arbitrarily large, this sum approaches the two-dimensional integral over all k, so that in the limit we obtain:

$$\langle\omega(\underline{r})\omega(\underline{r}')\rangle \propto \int d^2k \, \frac{k^2}{k^2+\kappa^2} \exp[i\underline{k}\cdot(\underline{r}-\underline{r}')]$$

$$\propto \int d^2k \left(1 - \frac{\kappa^2}{k^2+\kappa^2}\right) \exp[i\underline{k}\cdot(\underline{r}-\underline{r}')]$$

$$\propto \delta^{(2)}(\underline{r}-\underline{r}') + \kappa^2 \, G(\underline{r},\underline{r}';\kappa), \tag{4}$$

where $\delta^{(2)}(\underline{r}-\underline{r}')$ represents the two-dimensional Dirac delta function. Noting that the Green's function, $G(\underline{r},\underline{r}';\kappa)$, is the solution of the differential equation,

$$(\nabla^2 - \kappa^2) \, G(\underline{r},\underline{r}';\kappa) = \delta^{(2)}(\underline{r}-\underline{r}')$$

that vanishes as $|\underline{r}-\underline{r}'| \to \infty$, we finally obtain:

$$\langle\omega(\underline{r})\omega(\underline{r}')\rangle \propto \delta^{(2)}(\underline{r}-\underline{r}') - \frac{\kappa^2}{2\pi} K_0(\kappa|\underline{r}-\underline{r}'|), \tag{5}$$

where K_0 is the modified Bessel function of the second kind.[8]

This seemingly innocent result is actually quite profound. The delta function, of course, reflects the equipartition of an infinite number of high-wavenumber degrees of freedom of our continuous Euler fluid and is the spatial representation of the classical Rayleigh-Jeans ultraviolet catastrophe.

The surprising feature of our result is that it is precisely the result that Montgomery obtained for the charge-density autocorrelation function from a BBGKY hierarchy calculation, truncated at second order, of a discretized version of the two-dimensional, electrostatic, guiding-center plasma model.[9] This model is isomorphic to the Euler fluid model that we have been discussing. Although the physics of the continous and the discrete models of the two-dimensional fluid, as well as the approximations employed in their analyses, are manifestly different, these statistical calculations lead to identical results! Thus the presence of the delta-function in eq. (5) is consistent with the physical interpretation that the continuous Euler fluid's mean configuration described by this equation has a discrete vortex nature.[10]

If one were to write down the three-dimensional version of the right-hand side of eq. (4), curiously one would obtain the standard Debye-Hückel formula for the autocorrelation of the fluctuations of the electron number density in an infinite, homogeneous, three-dimensional, electrostatic electron plasma that is in thermal equilibrium[11]:

$$\langle \delta n(\underset{\sim}{r}) \delta n(\underset{\sim}{r}') \rangle \propto \delta^{(3)}(\underset{\sim}{r} - \underset{\sim}{r}') - \frac{\kappa^2}{4\pi|\underset{\sim}{r} - \underset{\sim}{r}'|} \exp(-\kappa|\underset{\sim}{r} - \underset{\sim}{r}'|).$$

In this equation, $n(\underset{\sim}{r})$ represents the electron number density and κ, the inverse Debye length. This formula, which is physically valid when $|\underset{\sim}{r} - \underset{\sim}{r}'| > \kappa^{-1}$, is inaccurate in its short-range predictions because of the failure, in this domain, of the weak-coupling approximation used in its derivation.

We have belabored this discussion in the attempt to delineate two points:

(1) Dynamical understanding of short-range correlations (high-wavenumber spectrum) has no necessary bearing on the understanding of either the longer-range correlations, $|\underset{\sim}{r} - \underset{\sim}{r}'| > \kappa^{-1}$, or, equivalently, the low-wavenumber spectrum, $k < \kappa$.

(2) The squared length, κ^{-2}, is merely the ratio of two inverse "temperatures."

In summary, we believe that if dissipative effects are sufficiently small, classical statistical mechanics of ideal systems may provide a credible approximation of both a turbulent spectrum when $k < \kappa$, and a two-point correlation function when $|\underset{\sim}{r} - \underset{\sim}{r}'| > \kappa^{-1}$. One may not need a prior understanding of the effects of dissipation on the high-wavenumber region of the spectrum.

4. APPLICATION TO A THREE-DIMENSIONAL, BOUNDED MAGNETOFLUID

We now shall apply the algorithm to a uniform-density, three-dimensional, incompressible magnetofluid bounded by an infinitely long, circular-cross-section, perfectly conducting wall, of radius r_0, concentric with the z-axis. An axial periodicity length of L is assumed for the magnetofluid. In the next section, this analysis will lead to our finding that the mean magnetic field, $\langle \underset{\sim}{B} \rangle$, satisfies the force-free equation:

$$\nabla \times \langle \underset{\sim}{B} \rangle = \mu \langle \underset{\sim}{B} \rangle, \qquad \Theta = \frac{|\mu| r_0}{2};$$

where μ represents the ratio of the inverse "temperature" associated with the magnetic helicity to that associated with the energy. The discussion of this section will also prepare us for the next section's treatment of the autocorrelation tensor of the field fluctuations, $\langle \delta \underset{\sim}{B}(\underset{\sim}{r}) \delta \underset{\sim}{B}(\underset{\sim}{r}') \rangle$.

The dynamical equations describing the evolution of the magnetofluid are:

$$\rho_0 \left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} \right) = -\nabla p + \frac{1}{\mu_p} (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}, \qquad \nabla \cdot \underset{\sim}{v} = 0; \tag{6a}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}), \qquad \nabla \cdot \underset{\sim}{B} = 0; \tag{6b}$$

where μ_p is the magnetic permeability of free space and bears no relation to the parameter μ. The required boundary conditions are $\underset{\sim}{B} \cdot \hat{n}|_{r_0} = \underset{\sim}{v} \cdot \hat{n}|_{r_0} = 0$.

The magnetic and velocity fields can be expanded in an appropriate complete set of basis functions.[2] The dynamical equations then can be shown to yield a Liouville theorem. Since the details will be presented elsewhere,[6] we shall omit them and take the heuristically direct path to the salient results.

There are three quadratic invariants of eqs. (6a,b), given the assumed boundary conditions; namely, the energy:

$$W \equiv W_B + W_K, \qquad W_B \equiv \frac{1}{2\mu_p}\int B^2 \frac{d^3r}{2\pi L}, \qquad W_K \equiv \frac{\rho_0}{2}\int v^2 \frac{d^3r}{2\pi L}; \tag{7}$$

the cross helicity:

$$H_c \equiv \int \underset{\sim}{v}\cdot\underset{\sim}{B}\, \frac{d^3r}{2\pi L};$$

and the magnetic helicity:

$$K \equiv \frac{1}{2\mu_p}\int \underset{\sim}{A}\cdot\underset{\sim}{B}\, \frac{d^3r}{2\pi L}. \tag{8}$$

For a prescribed magnetic field, $\underset{\sim}{B}(\underset{\sim}{r},t)$, bearing a conserved axial magnetic flux, $\pi r_0{}^2 B_0$, we can specify uniquely the vector potential in the Coulomb gauge by the condition that $\nabla \times \underset{\sim}{A}(\underset{\sim}{r},t)$ be the prescribed field and that

$$\underset{\sim}{A}(\underset{\sim}{r},t) = B_0\frac{r}{2}\hat{\theta} + \underset{\sim}{A}^J(\underset{\sim}{r},t), \tag{9}$$

where

$$\underset{\sim}{A}^J \times \hat{n}|_{r_0} = 0. \tag{10}$$

The first term on the right-hside of eq. (9) originates from the presence of a net axial magnetic flux; the magnetically fluxless, second term originates from the presence of current density in the magnetofluid. Equations (9) and (10) guarantee that the magnetic helicity integral, eq. (8), is a measure of the knottedness (or self-linkage) of the magnetic field within the magnetofluid.[5,6]

To expand $\underset{\sim}{A}^J$, we shall use the orthonormal set of fluxless eigenvectors of the curl operator:

$$\nabla \times \underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r}) = \mu_{m\ell n}\underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r}), \tag{11}$$

that satisfy:

$$\underset{\sim}{\xi}^{m\ell n}\cdot\hat{n}|_{r_0} = 0. \tag{12}$$

Following Chandrasekhar and Kendall,[12] we generate these eigenvectors from the scalar solutions, $J_m(\alpha_{m\ell n}r)\exp[i(m\theta + k_\ell z)]$, of the Helmholtz equation,

$$(\nabla^2 + \mu^2_{m\ell n})\phi^{m\ell n}(\underset{\sim}{r}) = 0,$$

where $\alpha_{m\ell n} = (\mu^2_{m\ell n} - k^2_\ell)^{1/2}$ and $k_\ell = 2\pi\ell/L$. One can verify that the required eigenvectors are given by:

$$\underset{\sim}{\xi}^{m\ell n} = N_{m\ell n}[\mu_{m\ell n}\nabla \times \phi^{m\ell n}\hat{z} + \nabla \times (\nabla \times \phi^{m\ell n}\hat{z})], \tag{13}$$

where $\hat{z}$ is the unit axial vector and $N_{m\ell n}$ is the normalization constant that guarantees:

$$\int \underset{\sim}{\xi}^{*m\ell n}(\underset{\sim}{r})\cdot\underset{\sim}{\xi}^{m'\ell'n'}(\underset{\sim}{r})\,\frac{d^3r}{2\pi L} = \delta_{mm'}\delta_{\ell\ell'}\delta_{nn'}. \tag{14}$$

Our integration domain for the cylinder is always taken to be $0 \leq \theta < 2\pi$, $0 \leq z < L$, $0 \leq r \leq r_0$. The boundary condition, eq. (12), which is trivially satisfied when $m = \ell = 0$, yields

$$m(\mu_{m\ell n}r_0)J_m(\alpha_{m\ell n}r_0) + (k_\ell r_0)(\alpha_{m\ell n}r_0)J'_m(\alpha_{m\ell n}r_0) = 0 \tag{15}$$

when $m^2 + \ell^2 > 0$. When m and ℓ are simultaneously zero, the condition of zero flux requires that

$$J_1(|\mu_{00n}|r_0) = 0. \tag{16}$$

We then find that $\underset{\sim}{A}^J(\underset{\sim}{r},t)$ has the solenoidal expansion:

$$\underset{\sim}{A}^J(\underset{\sim}{r},t) = \sum_{m\ell n} c_{m\ell n}(t)[\underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r}) - \nabla\zeta^{m\ell n}(\underset{\sim}{r})], \tag{17}$$

where the solution for $\zeta^{m\ell n}$ immediately follows from the solenoidal constraint, $\nabla^2\zeta^{m\ell n}(\underset{\sim}{r}) = 0$, regularity at the origin, and the boundary condition, eq. (10):

$$\zeta^{m\ell n}(\underset{\sim}{r}) = -\frac{i}{k_\ell}\left[\frac{N_{m\ell n}\alpha^2_{m\ell n}J_m(\alpha_{m\ell n}r_0)}{I_m(|k_\ell|r_0)}\right] I_m(|k_\ell|r)\exp[i(m\theta + k_\ell z)], \qquad \ell \neq 0;$$

$$\zeta^{m0n}(\underset{\sim}{r}) = \frac{iN_{m0n}\mu^2_{m0n}J'_m(|\mu_{m0n}|r_0)}{mr_0{}^{|m|-1}} r^{|m|}\exp(im\theta), \qquad m \neq 0;$$

$$\zeta^{00n}(\underset{\sim}{r}) = N_{00n}\mu^2_{00n}J_0(|\mu_{00n}|r_0)z;$$

where I_m represents the modified Bessel function of the first kind.[8] Reality of $\underset{\sim}{A}$ demands that $c_{m\ell n}(t) = c^*_{-m-\ell n}$.

Given this expansion of $\underset{\sim}{A}$, we shall derive the expansion of the magnetic field, $\underset{\sim}{B}(\underset{\sim}{r},t)$, where

$$\underset{\sim}{B}(\underset{\sim}{r},t) = B_0\hat{z} + \underset{\sim}{B}^J(\underset{\sim}{r},t).$$

The total, time-independent, axial magnetic flux of $\underset{\sim}{B}(\underset{\sim}{r},t)$ is borne by $B_0\hat{z}$. The fluxless second component, $\underset{\sim}{B}^J(\underset{\sim}{r},t)$, arises only from the presence of current density in the magnetofluid. We shall suppose that:

$$\underset{\sim}{B}^J(\underset{\sim}{r},t) \equiv \nabla \times \underset{\sim}{A}^J(\underset{\sim}{r},t) = \sum_{m\ell n} d_{m\ell n}(t)\mu_{m\ell n}\underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r}).$$

Using eqs. (11) and (14) and integrating by parts, we find

$$\mu_{m\ell n}d_{m\ell n}(t) = \int \nabla\cdot[\underset{\sim}{A}^J(\underset{\sim}{r},t) \times \underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r})]\,\frac{d^3r}{2\pi L} + \int \mu_{m\ell n}\underset{\sim}{A}^J(\underset{\sim}{r},t)\cdot\underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r})\,\frac{d^3r}{2\pi L}.$$

Equation (10) ensures the vanishing of the first integral. Using the expansion of $\underset{\sim}{A}^J$ - eq. (17), the boundary condition - eq. (12), and the fluxlessness of the $\underset{\sim}{\xi}^{m\ell n}$'s, we obtain the result, $d_{m\ell n}(t) = c_{m\ell n}(t)$. Thus, we conclude that

$$\underset{\sim}{B}(\underset{\sim}{r},t) = B_0\hat{z} + \sum_{m\ell n} c_{m\ell n}(t)\mu_{m\ell n}\underset{\sim}{\xi}^{m\ell n}(\underset{\sim}{r}). \qquad (18)$$

In contrast, one should be careful to observe that the expansion of the current density distribution is not generally obtainable from term-by-term curling of the expansion, eq. (18).[6]

By virtue of our algorithm, the distribution of our absolute equilibrium ensemble is proportional to $\exp[-2\mu_p(W - \mu K)/\epsilon - H_c/\tau]$. For simplicity, we shall consider only the case $\tau = \infty$, in which the velocity dependence can be factored out of the distribution and therefore has no effect on means values of functions that are solely magnetic-field dependent. We thus need to consider only the distribution,

$$D_{eq}(\underset{\sim}{\Gamma}) \propto \exp\left[-\frac{2\mu_p}{\epsilon}\,[W_B(\underset{\sim}{\Gamma}) - \mu\, K(\underset{\sim}{\Gamma})]\right], \qquad \epsilon > 0; \qquad (19)$$

where $\underset{\sim}{\Gamma}$ is the infinite-dimensional vector formed from the linearly independent, real and imaginary parts of the spectral coefficients, $\{c_{m\ell n}\}$, that completely specify the magnetic field configuration. The mean value of any quantity, $Q(\underline{\Gamma})$, is then taken to be the ensemble-averaged value:

$$\langle Q(\underline{\Gamma})\rangle = \frac{\int Q(\underline{\Gamma})\, D_{eq}(\underline{\Gamma})\, d\underline{\Gamma}}{\int D_{eq}(\underline{\Gamma})\, d\underline{\Gamma}},$$

where the "phase space" integration over $\underline{\Gamma}$ is defined to be integration from $-\infty$ to $+\infty$ for each of the linearly independent, real and imaginary parts of the spectral coefficients.

To obtain the functional dependence of $D_{eq}(\underline{\Gamma})$, we merely insert the expansions, eqs. (9), (17), and (18), into the expressions for W_B and K, eqs. (7) and (8), respectively. Using the boundary condition on the eigenvectors of the curl operator - eq. (12), their fluxlessness, and their mutual orthonormality - eq. (14), we obtain:

$$W_B = \frac{1}{2\mu_p}\Big[\frac{(B_0 r_0)^2}{2} + \sum_{m\ell n} |c_{m\ell n}|^2 \mu_{m\ell n}^2\Big],$$

$$K = \frac{1}{2\mu_p}\Big[-B_0 r_0 \sum_{n} c_{00n} + \sum_{m\ell n} \mu_{m\ell n}|c_{m\ell n}|^2\Big].$$

Hence, the distribution has the structure:

$$\exp\Big\{-\frac{1}{\varepsilon}\sum_{n}\big[c_{00n}^2(\mu_{00n}^2 - \mu\mu_{00n}) + \mu B_0 r_0 c_{00n}\big] - \frac{1}{\varepsilon}\sum_{m\ell n}\big[(\mu_{m\ell n}^2 - \mu\mu_{m\ell n})|c_{m\ell n}|^2\big]\Big\},$$

where only states satisfying $m^2 + \ell^2 > 0$ are included in the second summation. We then find that

$$\langle c_{00n}\rangle = -\frac{\mu r_0}{2(\mu_{00n}^2 - \mu\mu_{00n})} B_0 \tag{20}$$

and

$$\langle \delta c_{m\ell n}\delta c_{m\ell n}^*\rangle = \frac{\varepsilon}{2(\mu_{m\ell n}^2 - \mu\mu_{m\ell n})} \tag{21}$$

are the only nonvanishing mean values both of the coefficents and of the quadratic products of the coefficents' fluctuations, where:

$$\delta c_{m\ell n} \equiv c_{m\ell n} - \langle c_{m\ell n}\rangle .$$

Note that both the integrability of the distribution function and the positivity of all the $\langle|\delta c_{m\ell n}|^2\rangle$ requires that $|\mu|$ be less than the smallest positive eigenvalue of the set $\{\mu_{m\ell n}\}$, which we shall call μ_{min}. In the limit of large aspect ratio, L/r_0, Taylor has shown that μ_{min} occurs at approximately $3.11/r_0$ when $|m| = 1$ and $|k_\ell|r_0 \approx 1.23$. The relative signs of m, k_ℓ and μ are governed by the constraint:

$$\mathrm{sgn}(mk_\ell\mu) = +. \tag{22}$$

Using eq. (21), we can evaluate the mean magnetic energy and the mean magnetic helicity present in the magnetic fluctuations about the mean field:

$$\langle \delta W_B\rangle = \frac{1}{2\mu_p}\sum_{m\ell n}\langle|\delta c_{m\ell n}|^2\rangle\mu_{m\ell n}^2 = \frac{\varepsilon}{4\mu_p}\sum_{m\ell n}\frac{\mu_{m\ell n}}{\mu_{m\ell n} - \mu}, \tag{23a}$$

$$\langle \delta K \rangle = \frac{\varepsilon}{4\mu_p} \sum_{m\ell n} \frac{1}{\mu_{m\ell n} - \mu}, \qquad |\mu| < \mu_{min}. \tag{23b}$$

We note from eqs. (15) and (16) that each eigenvalue that occurs in the sum over states can be paired off with an oppositely signed counterpart. As a result, we find that asymptotically, for large $\mu_{m\ell n}$, each pair contributes a term in $\langle \delta K \rangle$ that is proportional to $\mu/\mu^2_{m\ell n}$; whereas the corresponding term in $\langle \delta W_B \rangle$ is asymptotically constant. This behavior of the energy spectrum of the magnetic fluctuations, characteristic of the Rayleigh-Jeans ultraviolet catastrophe in a continuous system, is vitiated in a physical plasma by dissipative effects. We thus obtain a heuristic understanding of the enhanced decay of magnetic energy with respect to the decay of magnetic helicity in a turbulent, dissipative magnetofluid that is bounded by a perfectly conducting wall.

Having noted that the presence of the two-dimensional Euler fluid energy led to the analogue of an inverse Debye length, κ, allowing for reasonable spectral predictions for $k < \kappa$, we surmise that the presence of magnetic helicity in our three-dimensional magnetohydrodynamic model may lead to an analogue of the inverse Debye length; namely, μ. We are thus suggesting that the ideal model may provide a reasonable description of physical data taken from reversed-field-pinch experiments having sufficiently low dissipation when such data consist of two-point correlations satisfying $|\underset{\sim}{r} - \underset{\sim}{r}'| > \mu^{-1}$, or equivalently, when such data refer to spectral modes whose wavenumbers are less than or of the order of μ.

Finally, one should note from eqs. (23a,b) that when $|\mu| < \mu_{min}$ and $\varepsilon \neq 0$, the fluctuation spectrum contains a broad band of spectral contributions. However as $\mu \to \mu_{min}$ and $\varepsilon \to 0$, such that

$$\frac{\varepsilon}{\mu_{min} - \mu} \to 4\mu_p \langle \delta K \rangle,$$

the significant region of the spectrum narrows. We find that only one spectral mode, the $m = 1$ mode that corresponds to the value $\mu = \mu_{min}$ discussed earlier in this section, yields a finite contribution to both the helicity and energy of the magnetic fluctuations. (A similar result occurs if $\mu \to -\mu_{min}$.) In addition, the energy generally receives a finite contribution emanating from infinitesimal contributions of a large number of high-wavenumber modes, the details of which await an understanding of the dynamical effects of dissipation. If $\langle\delta W\rangle = \mu_{min}\langle\delta K\rangle$, then this high-wavenumber contribution would be absent.

This spectral narrowing as both $\varepsilon \to 0$ and $\mu \to \mu_{min}$ (which we shall note in the next section corresponds to $\Theta \to 1.56$) may be related to the quiet behavior at $\Theta \approx 1.5$-1.6 observed in the ZT-40M and HBTX-1A experiments.

5. THE MEAN FIELD AND THE FIELD-FLUCTUATION AUTOCORRELATIONS

In this section, we shall consider the effects of the $m = \ell = 0$ modes alone. From eq. (20), we observe that these modes are the only ones needed for evaluating the mean value of $\underset{\sim}{B}$. Although they are insufficient for the complete evaluation of the autocorrelation tensor, $\langle\delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle$, which requires a much lengthier presentation, we shall nevertheless evaluate their contribution to $\langle\delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle$ in order to demonstrate a few of the techniques required in the complete analysis.[6]

For convenience, we shall begin by defining a new notation for efficient treatment of the $m = \ell = 0$ eigenvectors and eigenvalues. These eigenvectors satisfy:

$$\nabla \times \underset{\sim}{\xi}_n^{\pm}(r) = \pm\mu_n\underset{\sim}{\xi}_n^{\pm}(r), \qquad \mu_n > 0, \qquad n = 1,2,\cdots, \tag{24}$$

which has the solutions, orthonormal with respect to $\int_0^{r_0} r dr$:

$$\underset{\sim}{\xi}_n^{\pm}(r) = \frac{\pm J_1(\mu_n r)\hat{\theta} + J_0(\mu_n r)\hat{z}}{r_0 J_0(\mu_n r_0)}, \tag{25}$$

where $\hat{\theta}$ is the unit vector in the azimuthal direction. The condition,

$$J_1(\mu_n r_0) = 0,$$

guarantees their fluxlessness.

Using eqs. (18) and (24), we note that $\underset{\sim}{B}(\underset{\sim}{r})$ can be expanded as follows, using the new notation:

$$\langle \underset{\sim}{B}(\underset{\sim}{r})\rangle = B_0\hat{z} + \sum_{n=1}^{\infty} \mu_n \, [\langle c_n^+\rangle \underset{\sim}{\xi}_n^+(r) - \langle c_n^-\rangle \underset{\sim}{\xi}_n^-(r)], \tag{26}$$

where, via eq. (20),

$$\langle c_n^{\pm}\rangle = - \left[\frac{\mu r_0}{2(\mu_n^2 \mp \mu\mu_n)}\right] B_0. \tag{27}$$

Inserting eqs. (25) and (27) into eq. (26), we conclude that the infinite sum converges to:[13]

$$\langle \underset{\sim}{B}(\underset{\sim}{r})\rangle = \frac{B_0}{2} \left[\frac{|\mu| r_0}{J_1(|\mu| r_0)}\right][\mathrm{sgn}(\mu) J_1(|\mu| r)\hat{\theta} + J_0(|\mu| r)\hat{z}], \tag{28}$$

whose total magnetic flux is, of course, $\pi r_0{}^2 B_0$. If one calculates the Θ parameter of this mean field, one finds that

$$\Theta = \frac{|\mu| r_0}{2}.$$

Recognizing that the mean magnetic field for the ensemble distribution specified by eq. (19) also must be the most probable field, one sees that $\langle \underset{\sim}{B}(\underset{\sim}{r})\rangle$ must be the solution of the fixed-flux variational problem:

$$\delta[W_B - \mu K] = 0, \qquad \text{flux} = \pi r_0{}^2 B_0.$$

This equation is precisely the mathematical formulation of Taylor's minimum energy principle.

The value of our ensemble distribution, eq. (19), is that it permits calculations of expectation values involving higher moments of fields. We shall now utilize it to calculate the m = ℓ = 0 contribution to the field-fluctuation autocorrelation tensor.

We observe that

$$\langle \delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle|_{m=\ell=0} = \sum_{n=1}^{\infty} \mu_n^2[\langle(\delta c_n^+)^2\rangle \underset{\sim}{\xi}_n^+(r)\underset{\sim}{\xi}_n^+(r') + \langle(\delta c_n^-)^2\rangle \underset{\sim}{\xi}_n^-(r)\underset{\sim}{\xi}_n^-(r')]$$

$$= \frac{\varepsilon}{2}\sum_{n=1}^{\infty} [\underset{\sim}{\xi}_n^+(r)\underset{\sim}{\xi}_n^+(r') + \underset{\sim}{\xi}_n^-(r)\underset{\sim}{\xi}_n^-(r')] + \frac{\mu\varepsilon}{2}\sum_{n=1}^{\infty} [\frac{\underset{\sim}{\xi}_n^+(r)\underset{\sim}{\xi}_n^+(r')}{\mu_n - \mu} - \frac{\underset{\sim}{\xi}_n^-(r)\underset{\sim}{\xi}_n^-(r')}{\mu_n + \mu}],$$

where we have used, via eq. (21),

$$\frac{\varepsilon}{2(\mu_n^2 \mp \mu\mu_n)}$$

for the ensemble-averaged value of $(\delta c_n^{\pm})^2$.

If we separate the θ- and z-components and introduce the convention that $\mu_0 \equiv 0$, we obtain:

$$\langle\delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle|_{m=\ell=0} = \frac{\varepsilon}{r_0{}^2}\sum_{n=1}^{\infty}\{\frac{J_1(\mu_n r)J_1(\mu_n r')}{[J_0(\mu_n r_0)]^2}\}\hat{\theta}\hat{\theta} + \frac{\varepsilon}{r_0{}^2}\sum_{n=0}^{\infty}\{\frac{J_0(\mu_n r)J_0(\mu_n r')}{[J_0(\mu_n r_0)]^2}\}\hat{z}\hat{z}$$

$$+ \frac{\varepsilon}{r_0{}^2}\sum_{n=1}^{\infty}(\frac{\mu^2}{\mu_n^2 - \mu^2})\{\frac{J_1(\mu_n r)J_1(\mu_n r')}{[J_0(\mu_n r_0)]^2}\}\hat{\theta}\hat{\theta} + \frac{\varepsilon}{r_0{}^2}\sum_{n=0}^{\infty}(\frac{\mu^2}{\mu_n^2 - \mu^2})\{\frac{J_0(\mu_n r)J_0(\mu_n r')}{[J_0(\mu_n r_0)]^2}\}\hat{z}\hat{z}$$

$$+ \frac{\varepsilon}{r_0{}^2}\sum_{n=0}^{\infty}(\frac{\mu\mu_n}{\mu_n^2 - \mu^2})\{\frac{J_1(\mu_n r)J_0(\mu_n r')\hat{\theta}\hat{z} + J_0(\mu_n r)J_1(\mu_n r')\hat{z}\hat{\theta}}{[J_0(\mu_n r_0)]^2}\}. \qquad (29)$$

From the observation that each set of functions, $\{2^{1/2}J_0(\mu_n r)/r_0 J_0(\mu_n r_0)\}$ and $\{2^{1/2}J_1(\mu_n r)/r_0 J_0(\mu_n r_0)\}$, provides a complete, orthonormal set of basis functions, the sum of the first two sums in eq. (29) immediately follows:

$$\frac{\varepsilon\ \delta(r - r')}{2r}(\underset{\sim}{I} - \hat{r}\hat{r}),$$

where $\underset{\sim}{I}$ is the identity matrix. Similarly, the third and fourth sums can be seen to be related to the solutions of the Green's equation:

$$(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \mu^2)\, G_m(r,r';\mu) = -\frac{\delta(r - r')}{r},$$

with m = 1 and 0, respectively. The desired solutions are regular at the origin. The first must vanish at the boundary, whereas the radial derivative of the second must vanish at the boundary. Solving for G_1 leads to the result for the third sum of eq. (29):

$$\frac{\varepsilon\pi(\mu r_0)^2}{4r_0{}^2J_0(|\mu|r_0)}\underset{\sim}{T}(r,\mu)\underset{\sim}{T}(r',\mu) - \frac{\varepsilon|\mu r_0|^3 J_0(|\mu|r_0)}{2r_0{}^2J_1(|\mu|r_0)}\underset{\sim}{T}_\mu(r)\underset{\sim}{T}_\mu(r'),$$

where

$$\underset{\sim}{T}(r_<,\mu) \equiv J_1(|\mu|r_<)\hat{\theta},$$

$$\underset{\sim}{T}(r_>,\mu) \equiv [Y_0(|\mu|r_0)J_1(|\mu|r_>) - J_0(|\mu|r_0)Y_1(|\mu|r_>)]\hat{\theta},$$

$$\underset{\sim}{T}_\mu(r) \equiv -\frac{J_1(|\mu|r)}{(|\mu|r_0)J_0(|\mu|r_0)}\,\hat{\theta},$$

and where $r_<$ and $r_>$ respectively denote the lesser and greater of r and r′.

After evaluating G_0, we can perform the fourth summation of eq. (29) to obtain

$$-\frac{\varepsilon\pi(\mu r_0)^2}{4r_0{}^2J_1(|\mu|r_0)}\,\underset{\sim}{P}(r,\mu)\underset{\sim}{P}(r',\mu),$$

where

$$\underset{\sim}{P}(r_<,\mu) \equiv J_0(|\mu|r_<)\hat{z},$$

$$\underset{\sim}{P}(r_>,\mu) \equiv [J_1(|\mu|r_0)Y_0(|\mu|r_>) - Y_1(|\mu|r_0)J_0(|\mu|r_>)]\hat{z}.$$

The final sum can be extracted from the fourth sum by allowing the curl operator to act appropriately on the fourth term. If we define:

$$\mu\underset{\sim}{T}_A(r,\mu) \equiv \nabla \times \underset{\sim}{P}(r,\mu),$$

so that

$$\underset{\sim}{T}_A(r_<,\mu) = J_1(|\mu| r_<)\hat{\theta},$$

$$\underset{\sim}{T}_A(r_>,\mu) = [J_1(|\mu| r_0)Y_1(|\mu| r_>) - Y_1(|\mu| r_0)J_1(|\mu| r_>)]\hat{\theta},$$

then the final sum is expressible as:

$$-\frac{\varepsilon\pi(\mu r_0)^2\ \mathrm{sgn}(\mu)}{4r_0{}^2 J_1(|\mu| r_0)}\cdot[\underset{\sim}{T}_A(r,\mu)\underset{\sim}{P}(r',\mu) + \underset{\sim}{P}(r,\mu)\underset{\sim}{T}_A(r',\mu)].$$

If to these contributions to the field-fluctuation autocorrelation tensor are added the contributions from the $m^2 + \ell^2 \neq 0$ states, the complete autocorrelation tensor, $\langle\delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle$, is obtained.[6]

6. STRUCTURE OF THE QUIESCENT STATE

We shall analyze some properties of the theoretically "quiescent" state that forms as $\theta \to \mu_{min} r_0/2 \simeq 1.56$ and $\varepsilon \to 0$ (at $|k| r_0 = 1.23$), discussed in sec. 4. We shall assume that $\mu \to +\mu_{min}$. We then find, in accordance with the relative sign constraint, eq. (21), that

$$\langle\delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle \propto \mathrm{Re}[\underset{\sim}{\xi}_Q(\underset{\sim}{r})\underset{\sim}{\xi}_Q^*(\underset{\sim}{r}')] + \text{high-wavenumber terms}, \tag{30}$$

where $\underset{\sim}{\xi}_Q$ is the eigenfunction of the curl with $m = 1$, $k_\ell r_0 \simeq 1.23$, and $\mu r_0 \simeq 3.11$, and, of course, where Re is the reality operator.

When $\underset{\sim}{r} = \underset{\sim}{r}'$, let us consider the properties of the first term of eq. (30), which is a 3×3 symmetric matrix:

$$\mathrm{Re}[\underset{\sim}{\xi}_Q(\underset{\sim}{r})\underset{\sim}{\xi}_Q^*(\underset{\sim}{r}')].$$

Equation (13) demonstrates that, in the complex plane, the phase of the radial component of ξ_Q is shifted by $\pi/2$ from that of the azimuthal and axial components. Thus the $r\theta$, rz, θr, and zr-components of this matrix vanish. The determinant of this matrix vanishes because it is equal to the product of $|(\xi_Q)_r|^2$ with a vanishing determinant of a 2×2 symmetric submatrix. Therefore, the three orthogonal, principal axes are oriented such that one is in the $\hat{r}$-direction. The two nonvanishing eigenvalues, one associated with the radially-directed principal axis, define a plane in which the fluctuations occur. Computationally we find that the local normal to this plane is never more than about 8° from the local direction of the mean magnetic field (evaluated at $\Theta = 1.56$); i.e., the fluctuations contained in the first term of eq. (30) are approximately orthogonal to the mean magnetic field.

Although we do not generally anticipate valid results for $|\underset{\sim}{r} - \underset{\sim}{r}'| < \mu^{-1}$ due to the short-range effects of the high-wavenumber contributions, we can suppose that $\langle\delta W\rangle = \mu_{min}\langle\delta K\rangle$, which we have seen eliminates these contributions from the quiescent state. This narrowing of the spectrum down to a single $m = 1$ mode signals the onset of coherent $m = 1$ activity superimposed on a cylindrically-symmetric state. This superposition results in the helical force-free state, originally described by Taylor.[3]

7. STATUS OF THE STATISTICAL MODEL

The mathematical analysis of the complete autocorrelation tensor, $\langle\delta\underset{\sim}{B}(\underset{\sim}{r})\delta\underset{\sim}{B}(\underset{\sim}{r}')\rangle$, is detailed in ref. 6. We anticipate comparison of this tensor with experimental data as well as with data extracted from computer simulations of the dynamics of a bounded, three-dimensional magnetofluid.

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REFERENCES

1. M. R. C. Watts, private communication.

2. L. Turner, Phys. Rev. A 24 (1981) 2839.

3. J. B. Taylor, Phys. Rev. Lett. 33 (1974) 1139.

4. D. Montgomery, L. Turner, G. Vahala, Phys. Fluids 21 (1978) 757.

5. L. Turner and J. P. Christiansen, Phys. Fluids 24 (1981) 893.

6. L. Turner, "Statistical Mechanics of a Bounded, Ideal Magnetofluid," in preparation.

7. T. D. Lee, Quart. Appl. Math. 10 (1952) 69.

8. *Handbook of Mathematical Functions*, ed. by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington, D. C., 1965).

9. D. Montgomery, "Strongly Magnetized Classical Models," in *Plasma Physics*, Les Houches Lectures - 1972, ed. by C. DeWitt and J. Peyraud (Gordon and Breach, New York, 1975).

10. L. Onsager, Nuovo Cimento Suppl. 6, Series 9 (1949) 279.

11. A. G. Sitenko, *Electromagnetic Fluctuations in Plasma* (Academic Press, New York, 1967) p. 55.

12. S. Chandrasekhar and P. C. Kendall, Astrophys. J. 126 (1957) 457.

13. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1952) p. 381.